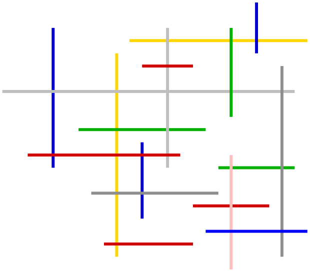
Intersection Graphs and Order Dimension

CG Week Workshop
Geometric Intersection Graphs
Eindhoven, 25.06.2015



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partly joint work with
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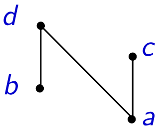
Outline

Linear Extensions

A linear extension of $P = (X, <)$ is a linear order L, such that

- $x <_P y \implies x <_L y$



d	c	d	d	c
c	d	b	c	d
b	b	c	a	a
a	a	a	b	b

Dimension of Orders I

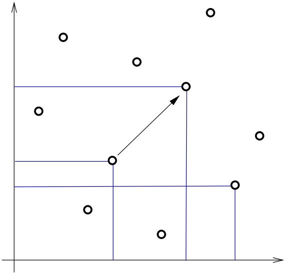
A family $\mathcal{L}$ of linear extensions is a realizer for $P = (X, <)$ provided that

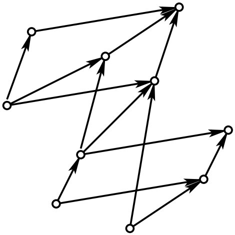
* for every incomparable pair (x, y) there is an $L \in \mathcal{L}$ such that $x < y$ in L.

The dimension, $\dim(P)$, of P is the minimum t, such that there is a realizer $\mathcal{L} = \{L_1, L_2 \ldots, L_t\}$ for P of size t.

Dimension of Orders II

The dimension of an order $P = (X, <)$ is the least t, such that P is isomorphic to a suborder of $\mathbb{R}^t$ with the product ordering.





Complexity

Theorem [Yannakakis 1982]. To test if a partial order has dimension $\leq k$ is NP-complete for all $k \geq 3$.
To test if a partial order of height 2 has dimension $\leq k$ is NP-complete for all $k \geq 4$.

Theorem [Chalermsook et al. 2013]. Unless NP = ZPP there is no polynomial algorithm to approximate the dimension of a partial order with a factor of $O(n^{1-\varepsilon})$ for any $\varepsilon > 0$

Containment

- Containment orders of intervals — dimension ≤ 2.
- Containment orders of triangles* — dimension ≤ 3.
- Containment orders of n-gons* — dimension $\leq n$.
- Containment orders of k-boxes — dimension $\leq 2k$.

*prescribed slopes

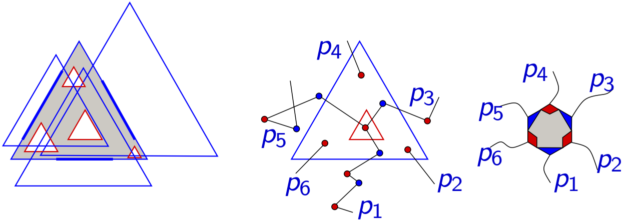
Complexity

Theorem [F., Mustaţă, and Pergel 2014]. To test if a partial order of height 2 has dimension 3 is NP-complete.

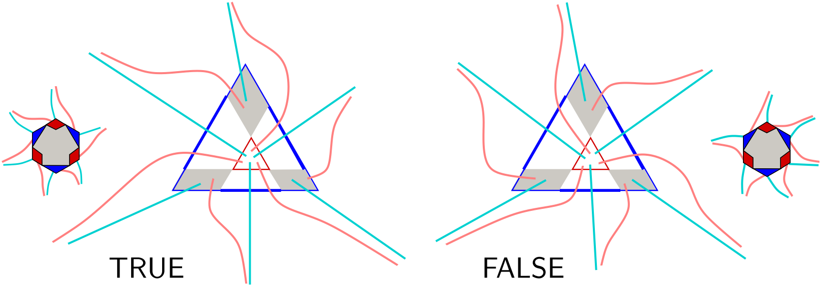
- The reduction is from planar 3-connected 3-SAT.
- For instance ϕ we construct a bipartite graph G_ϕ such that ϕ has a satisfying assingment if and only if G_ϕ has a triangle containment representation.

Basic Gadgets

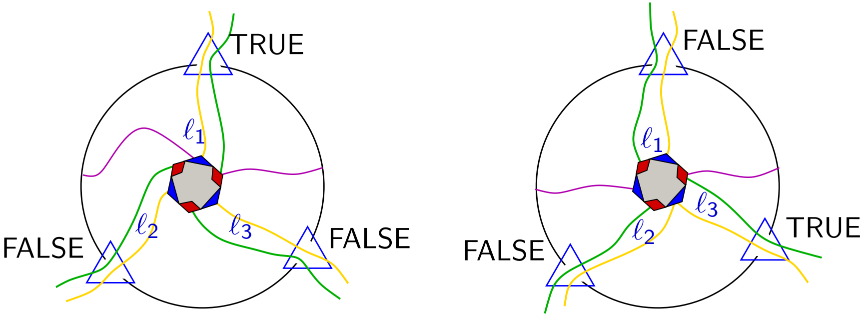
- *(occurrence gadget)* A pair of paths connecting clause and variable gadgets. They can not cross, sidedness transmits truth values.
- *(rotor gadgets)* Synchronize a sidedness choice.



Variable Gadget

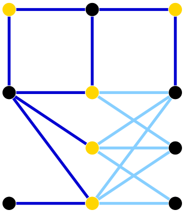


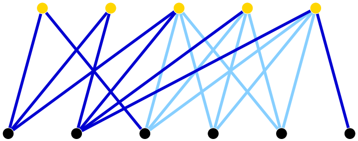
Clause Gadget



Bipartite Orders

A bipartite Graph can be viewed as a height 2 order.

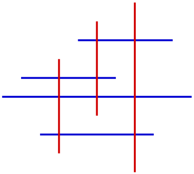


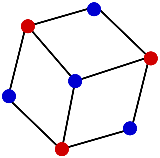


- We can talk about $\dim(G)$ when G is bipartite.

Grid Intersection Graphs

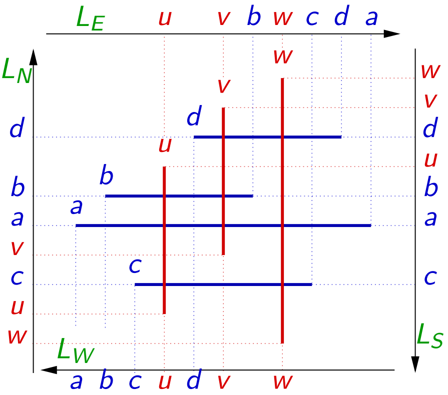
A GIG is an intersection graphs of horizontal and vertical segments.





- GIGs are bipartite.

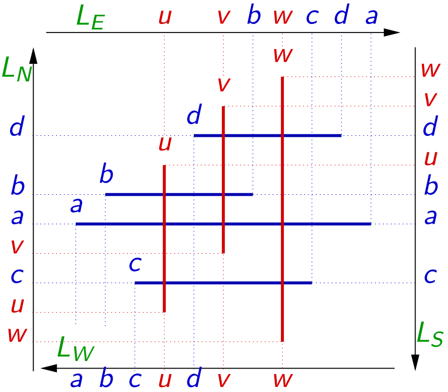
Dimension of GIGs



- In each projection minimals are taken early, maximals are taken late.

  Theorem. G a GIG, then $\dim(G) \leq 4$.

Dimension of GIGs

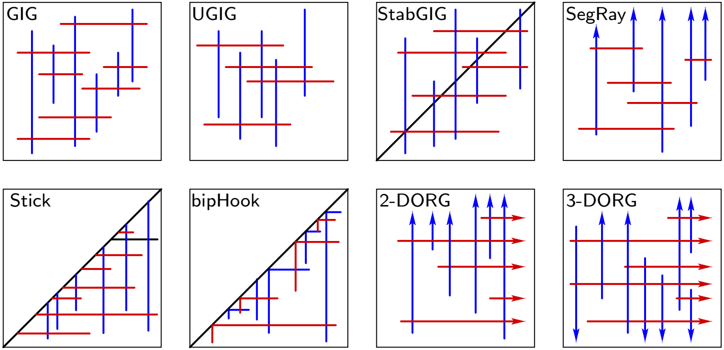


- In each projection minimals are taken early, maximals are taken late.

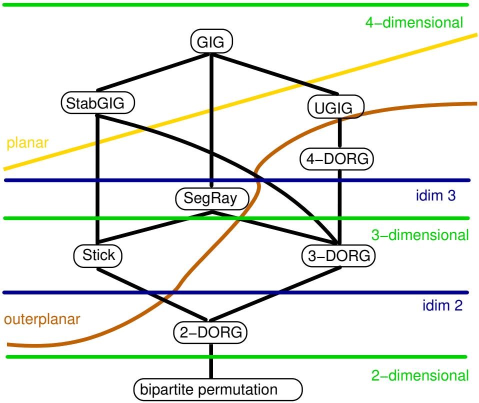
  Theorem. G a GIG, then $\dim(G) \leq 4$.

Theorem [generalization]. If a bipartite graph $G = (X, Y; E)$ has a representation as intersection graphs of objects from a t-separable class, then $\dim(G) \leq 2t$.

Subclasses of GIGs

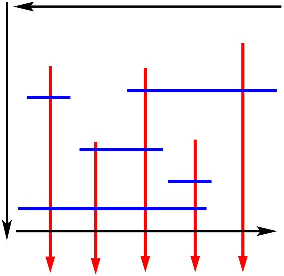


Subclasses of GIGs - The Inclusion Order

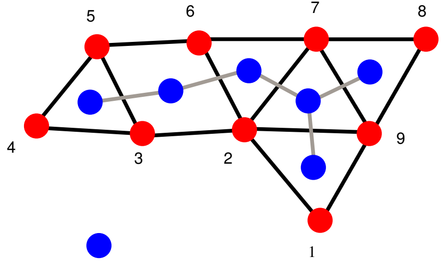


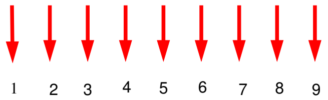
SegRay graphs

For interval dimension we only care of min-max pairs.
The interval dimension of a SegRay graph is at most 3.



Outerplanar vertex-face as SegRay





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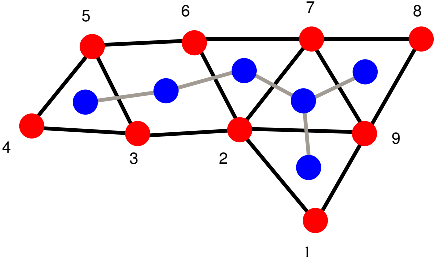
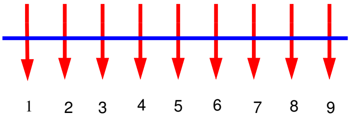
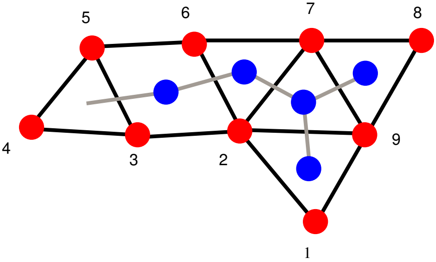
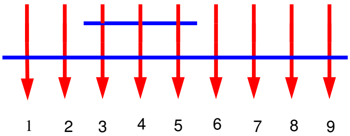




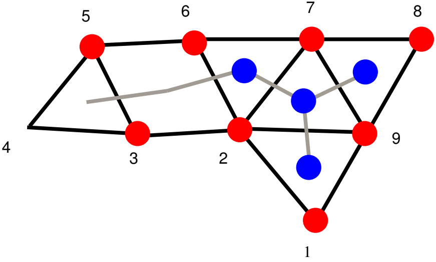
Outerplanar vertex-face as SegRay

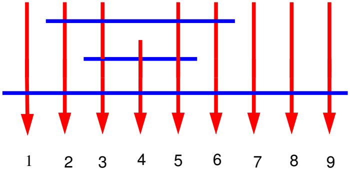
- Iterate using leaves of the dual tree



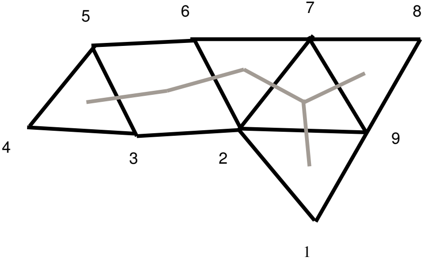


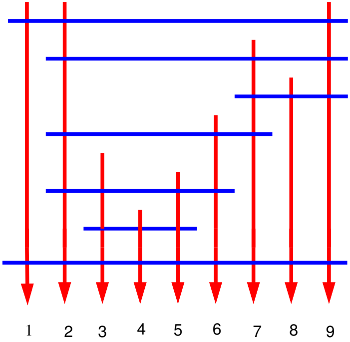
Outerplanar vertex-face as SegRay





Outerplanar vertex-face as SegRay





Dimension of SegRay graphs

- Vertex-face posets of outerplanar maps are SegRay graphs.

Theorem (F. and Nilsson 2006)

There are outerplanar maps with a 4-dimensional vertex-face poset.

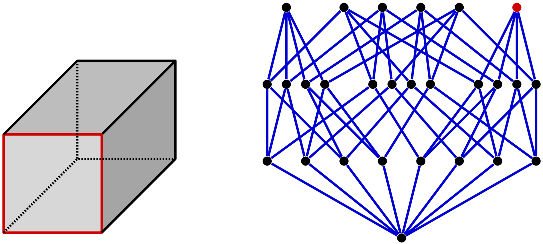
Corollary

There are SegRay graphs of dimension 4.

Corollary

The interval dimension of a vertex-face poset of an outerplanar map is 3.

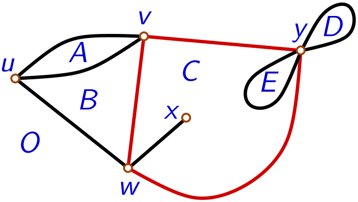
Brightwell–Trotter Theorem I

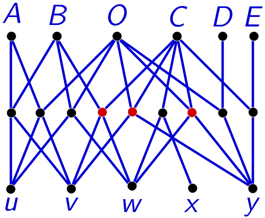


Theorem [Brightwell+Trotter '93].
If G is a 3-connected plane graph with a face F, then

- $\dim(P_{VEF}(G \setminus F)) = 3$
- $\dim(P_{VEF}(G)) = 4$

Brightwell–Trotter Theorem II





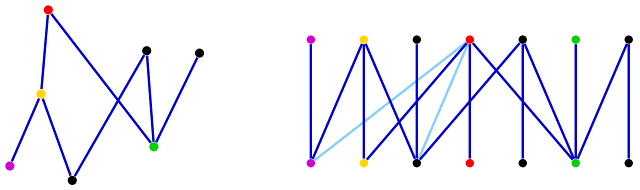
Theorem [Brightwell+Trotter '97].
If G is a plane multi-graph with loops, then

$$\dim(P_{VEF}(G)) \leq 4.$$

Splits and Dimension

The split of $P = (X, <)$ is $\text{split}(P) = (X' \cup X'', <_s)$ with

$$x' <_s y'' \quad \text{iff} \quad x \leq y$$

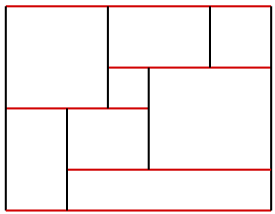


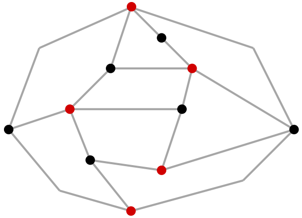
Theorem [Kimble 78].

$$\dim(P) \leq \dim(\text{split}(P)) \leq \dim(P) + 1.$$

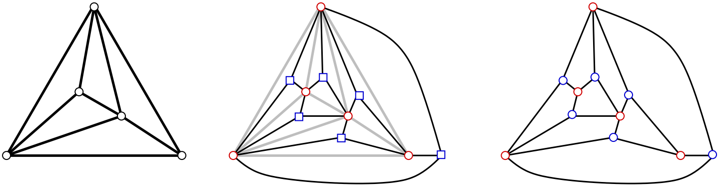
Planar Bipartite Graphs

Theorem [Hartman-Newman-Ziv '91 and de Fraysseix-Ossona de Mendez-Pach '95 **].**
Every planar bipartite graph H admits a contact representation with interiorly disjoint horizontal and vertical segments.





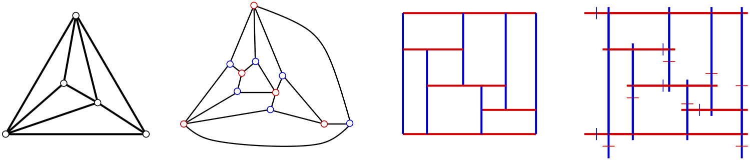
Angle Graphs of Planar Graphs



- Angle graphs are planar bipartite.
- They are the comparability graphs of vertex-face posets.

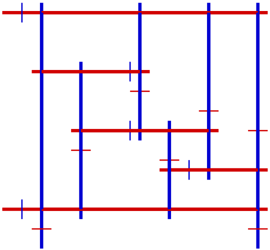
The First Step

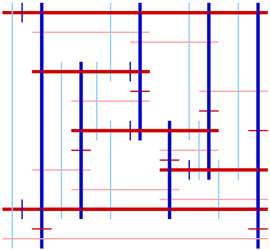
G 2-connected plane multi-graph (no loops).



- The order dimension of $P_{VF}(G)$, the incidence order of vertices and faces of a planar multigraph G (no loops) is at most four, moreover $\dim(\text{split}(P_{VF}(G)) \leq 4$.

Adding the Edges





Theorem. If G is a 2-connected and plane multigraph, then $\dim(\text{split}(P_{VEF}(G)) \leq 4$.

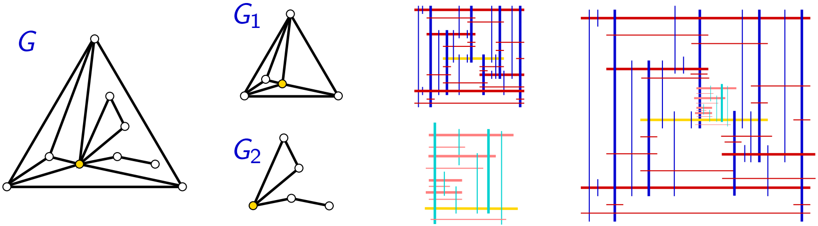
Loops

- Break loops by inserting a new vertex.

  $\mathrm{split}(P_{VEF}(G))$ is a suborder of $\mathrm{split}(P_{VEF}(G^+))$

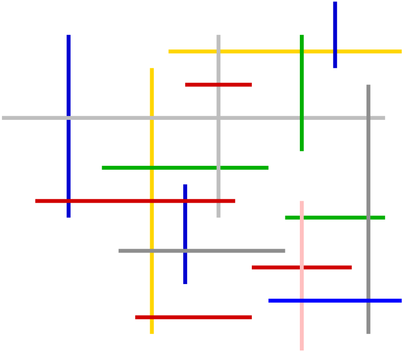
Cut Vertices

- Use induction: break G into G_1 and G_2 at a cut vertex:



Theorem. If G is a plane multigraph -loops allowed-, then $\dim(\text{split}(P_{VEF}(G)) \leq 4$.

The End



Thank You