Geometric Intersection Graphs: Problems and Directions
Collection of Open Problems

1. Coloring segment intersection graphs avoiding monochromatic cliques

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For a graph G and an integer $k \geq 2$, let $\chi_k(G)$ denote the minimum number of colors in a coloring of the vertices of G such that no k-clique of G is monochromatic. In particular, $\chi_2(G)$ is the ordinary chromatic number of G. It is known that triangle-free segment intersection graphs can have arbitrarily large chromatic number. It is also known that K_k-free string graphs can have arbitrarily large parameter χ_{k-1}, for $k \geq 3$, but the construction makes essential use of the fact that pairs of curves can cross arbitrarily many times [3]. Is it true that K_k-free segment intersection graphs can have the parameters $\chi_3, \chi_4, \ldots, \chi_{k-1}$ arbitrarily large?

This problem is related to a well-known conjecture that *k-quasi-planar geometric graphs*, that is, graphs drawn in the plane using straight-line edges no k of which pairwise cross, have $O(n)$ edges for every fixed k. The conjecture is known to be true for $k = 3$ [2] (with a simple proof) and $k = 4$ [1] (with a complicated proof), but it is open for $k \geq 5$. If there is a constant c_k such that $\chi_3(G) \leq c_k$ or $\chi_4(G) \leq c_k$ for every K_k-free segment intersection graph G, then the fact that the conjecture holds for 3 or 4 (respectively) implies that it holds for k [3].

References

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2. On-line coloring of 2-simple orders

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Originators: Stefan Felsner, Piotr Micek, Torsten Ueckerdt [2]

Imagine a graph that is incoming vertex by vertex where each new vertex is created with full adjacency status to previously created vertices. An on-line coloring algorithm colors each vertex when it is created, immediately and irrevocably, so that adjacent vertices receive distinct colors.

Imagine a graph is presented on-line with a geometric representation as a geometric intersection graph. Fix two horizontal lines in the plane and let the vertices of the incoming graph be curves spanned between these lines (that is, contained within the stripe in between and with endpoints on both lines). If the presented family is 1-simple (any two curves intersect at most once), then there is an on-line algorithm using $\binom{\chi+1}{2}$ colors, where χ is the chromatic number of the presented graph (see [3]).

What if the presented family of curves is 2-simple? The best known on-line algorithm follows from the on-line chain patitioning algorithm for posets and uses a superpolynomial number of colors in terms of the chromatic number (see [1]). The same upper bound is best known for families of 1-bend curves, that is, curves composed of two straight-line segments glued together. Some other interesting cases like triangle graphs, trapezoid graphs and others are studied in [2].

References

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3. Segment representations of co-planar graphs

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Originators: Jan Kratochvíl, Aleš Kuběna [3]

Co-planar graphs are the complements of planar graphs. In [3] Kratochvíl and Kuběna posed the question whether every co-planar graph can be represented as an intersection graph of segments. An original motivation for the question was that it would imply that the optimization problem Max Clique is NP-complete when restricted to segment graphs. In [1] it is shown that any planar graph has an even subdivision whose complement is a ray intersection graph, this implies that Max Clique is NP-complete for ray intersection graphs and hence for segment graphs.

Francis et al. [2] have shown that complements of partial 2-trees are segment graphs. More recently with P. Micek we found a representation of the complement of the grid graph $G(n,n)$ as intersection graph of pseudo-segments, i.e., as intersection graph of a family of curves in the plane such that any two curves have at most one point of intersection where they cross (no tangencies).

The general question remains unresolved. Here are two more specific questions:

- Is the complement of the grid graph $G(n,n)$ representable as intersection graph of segments?
- Is the triangular grid graph $T(n)$ representable as an intersection graph of pseudo-segments?

References

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4. Maximum clique for disks of two sizes

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We do not know how hard it is to find a largest clique in the intersection graph of disks. Here I would like to propose a more modest problem, hopefully easier.

Let G be the intersection graph of a family D of disks in the plane. Assume that D is given and contains disks of two different sizes. I am aware of two different algorithms to get a 2-approximation to the maximum clique problem in G. The first algorithm is to find the largest clique for each of the disk sizes independently using the algorithm by Clark, Colbourn and Johnson [2], and return the best. The second algorithm is to use the generic 2-approximation algorithm for arbitrary disks by Ambühl and Wagner [2]. Can we get a 1.99-approximation algorithm?

References

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5. Coloring and Independent Set of Rectangles

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I want to propose some problems related to computing the *stability number* and bounding the *chromatic number* of a rectangle intersection graph, when a rectangle representation is given. These are long-standing open problems in approximation algorithms, so improving the state of the art for them could be very hard. So I propose two related problems whose studies might be insightful for the purpose of attacking the original problems. These problems are of independent interest.

- **Finding a good cut sequence:** Let P be a subset of the plane. Let H_1 and H_2 be two halfplanes bounded by a straight line ℓ that is parallel to the x or y axis. Cutting P along line ℓ gives us two pieces $P_1 = P \cap H_1$ and $P_2 = P \cap H_2$. We say that a collection of non-overlapping rectangles $\mathcal{R}', |\mathcal{R}'| = m$, is *guillotine separable* if we can cut the plane into two parts and then successively cut each part into smaller pieces until we obtain m pieces, each containing precisely one of our m rectangles. Let $g(\mathcal{R})$ denote the minimum value β such that there is a subset $\mathcal{R}' \subseteq \mathcal{R}$ of size $|\mathcal{R}|/\beta$ that is guillotine separable. Pach and Tardos [3] proved that $g(\mathcal{R}) \leq O(\log |\mathcal{R}|)$ for all $\mathcal{R}$. Abed et al. [1] showed that $g(\mathcal{R}) \leq 81$ if $\mathcal{R}$ is a collection of *squares* and there is a collection of unit squares $\mathcal{R}$ for which $g(\mathcal{R}) \geq 2 - \epsilon$ for arbitrarily small ϵ.

  We remark that $g(\mathcal{R}) \leq \beta$ would imply a combinatorial β-approximation algorithm for computing stability number. Can we narrow down the gap between the lower bound of 2 and the upper bound of $O(\log n)$?
- **Coloring of hook graphs:** Consider a family of rectangles such that all top-left corners lie on the same line (says, $y = x$) in the plane. Can we show that $\chi(G) = O(\omega(G))$ for this family? It is known that $\chi(G) = O(\omega(G) \log \omega(G))$ [2] for this family. This problem has a strong connection to the integrality gap of an LP relaxation for *Unsplitable Flow problem (UFP)* on paths, a well-studied problem in combinatorial optimization.

  In fact, for the purpose of UFP integrality gap, it suffices to show that the fractional chromatic number $\chi_f(G)$ is at most $O(\omega(G))$.

References

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