Colorings of Geometric Intersection Graphs and Related Problems

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CG Week, 25.06.2015

Chromatic number, clique number, and χ-boundedness

For a graph G, we define

- the chromatic number χ — the minimum number of colors in a proper coloring of G
- the clique number ω — the maximum size of a clique in G

Obvious inequality: $\chi \geqslant \omega$

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Classical result

There exist graphs with $\omega = 2$ and χ arbitrarily large.

Theorem (Kim, 1995; Ajtai, Komlós, Szemerédi, 1980)

There exist graphs with $\omega = 2$ and $\chi = \Theta(\sqrt{n/\log n})$, and this is optimal.

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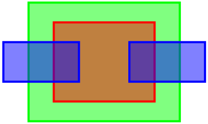
A class of graphs $\mathcal{G}$ is χ-bounded or if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every graph in $\mathcal{G}$ satisfies $\chi \leqslant f(\omega)$.

Examples will follow...

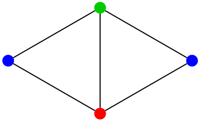
Intersection graphs and overlap graphs

For a family of sets $\mathcal{F}$, we define

- the intersection graph of $\mathcal{F}$ as the graph that has $\mathcal{F}$ as the set of vertices and the pairs of intersecting members of $\mathcal{F}$ as the set of edges



$\rightarrow$



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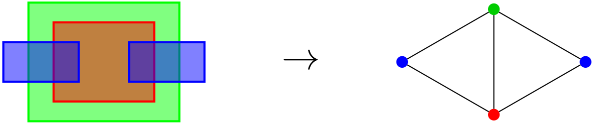
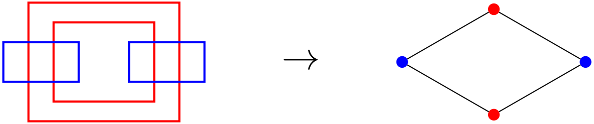
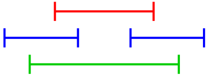
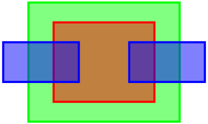


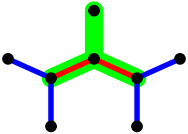
- the overlap graph of $\mathcal{F}$ as the graph that has $\mathcal{F}$ as the set of vertices and the pairs of overlapping (intersecting but not nested) members of $\mathcal{F}$ as the set of edges

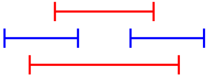


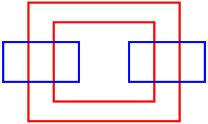
Geometric intersection graphs

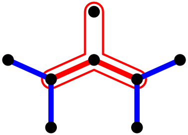

interval graphs

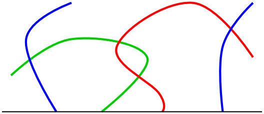

rectangle graphs

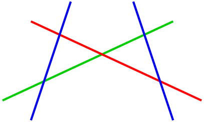

chordal graphs

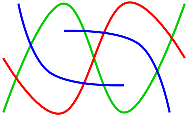

interval overlap graphs


rectangle overlap graphs

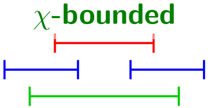

subtree overlap graphs


outerstring graphs

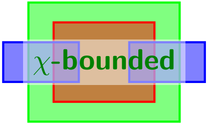

segment graphs


string graphs

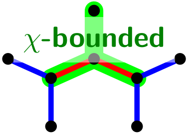
Geometric intersection graphs



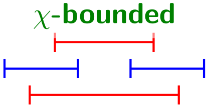
interval graphs



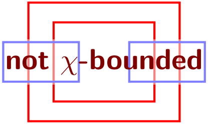
rectangle graphs



chordal graphs



interval overlap graphs



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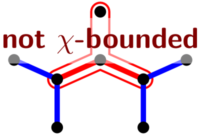
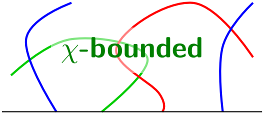
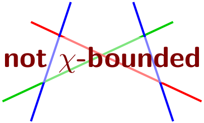


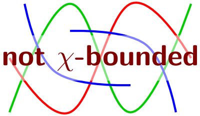
subtree overlap graphs



outerstring graphs



segment graphs



string graphs

Three types of χ-bounded classes

Perfect graphs — $\chi = \omega$:

- interval graphs
- chordal graphs ($=$ intersection graphs of subtrees of a tree)

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Graphs with average degree bounded by $f(\omega)$:

- circular-arc graphs (Tucker, 1975)
- intersection graphs of "fat" convex sets (Pach, 1980)
- geometric intersection graphs with excluded $K_{s,s}$ (Fox, Pach, 2010)

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Other χ-bounded classes of graphs:

- rectangle graphs (Asplund, Grünbaum, 1960)
- circle graphs ($=$ interval overlap graphs) (Gyárfás, 1985)
  and polygon-circle graphs (Kostochka, Kratochvíl, 1997)
- outerstring graphs (McGuinness, 1996 & 2000; Suk, 2014; Lasoń, Micek, Pawlik, W, 2014; Rok, W, 2014)

Bounds for χ-bounded classes

Rectangle graphs:

- upper bound: $\chi = O(\omega^2)$ (Asplund, Grünbaum, 1960; Hendler, 1998)
- upper bound, no containment: $\chi = O(\omega \log \omega)$ (Chalermsook, 2011)
- construction: $\chi = 3\omega$ (Kostochka, unpublished)

Circle graphs (= interval overlap graphs):

- upper bound: $\chi = O(2^\omega)$ (Kostochka, Kratochvíl, 1997; Černý, 2007)
- construction: $\chi = \Theta(\omega \log \omega)$ (Kostochka, 1988)

Problem: Improve the bounds.

Any promising approaches?

Classes that are not χ-bounded

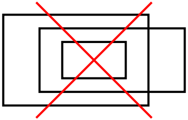
Rectangle overlap graphs:

- construction: $\omega = 2$, $\chi = \Theta(\log\log n)$ (Pawlik et al., 2013)
- upper bound: $\chi = O_\omega((\log\log n)^{\omega-1})$, clean: $\chi = O_\omega(\log\log n)$ (Krawczyk, Pawlik, W, 2013; Krawczyk, W, 2014)

Subtree overlap graphs:

- construction: $\chi = \Theta_\omega((\log\log n)^{\omega-1})$ (Krawczyk, W, 2014)
- upper bound: $\chi = O_\omega((\log\log n)^{\binom{\omega}{2}})$, clean: $\chi = O_\omega((\log\log n)^{\omega-1})$ (Krawczyk, W, 2014)

where an overlap graph is clean if its overlap model contains no three sets two of which overlap and both contain the third.



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Segment graphs, L-shape graphs, etc.:

- construction: $\omega = 2$, $\chi = \Theta(\log\log n)$ (Pawlik et al., 2013)
- upper bound: $\chi = O_\omega(\log n)$ (McGuinness, 1996 & 2000; Suk, 2014)

String graphs:

- construction: $\chi = \Theta_\omega((\log\log n)^{\omega-1})$ (Krawczyk, W, 2014)
- upper bound: $\chi = (\log n)^{O(\log\omega)}$ (Fox, Pach, 2014)

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Problem: Improve the asymptotic bounds.

Techniques

1. Induction
2. Decompositions into structures of some special kind
3. Connections to on-line coloring on tree-like structures
4. Reduction to graphs of radius 2 (McGuinness, 2001)
5. BFS and *k*-BFS decompositions (Gyárfás, 1985; Krawczyk, W, 2014)
6. χ-Boundedness of outerstring graphs
7. Planarity arguments

“Technical” problems

Theorem (McGuinness, 2001)

Every triangle-free 1-string graphs with large chromatic number contains an induced subgraph with radius 2 and with large chromatic number.

Problem

Is the same true for K_k-free 1-string graphs?

"Technical" problems

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Every triangle-free 1-string graphs with large chromatic number contains an induced subgraph with radius 2 and with large chromatic number.

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Theorem (Rok, W, 2014)
Outerstring graphs are χ-bounded.

Problem
Are intersection graphs of pseudo-parabolas (curves each crossing a given line in 2 points) χ-bounded?

This would be enough to conclude that intersection graphs of curves each crossing a given line in t points is χ-bounded.

Structure of graphs with large chromatic number

All known constructions of triangle-free geometric intersection graphs with chromatic number $\Theta(\log \log n)$ are esentially the same!
Call them the canonical construction.

Problem

Does every triangle-free segment intersection graph with chromatic number k contain a canonical construction of a graph with chromatic number $\Theta(k)$ as an induced subgraph?

Nothing like this can hold for graphs in general!

Variant of chromatic number

For a graph G, we define

- a K_k-free coloring — a coloring of vertices of G such that the subgraph of G induced by every color is K_k-free
- the K_k-free chromatic number χ_k — the minimum number of colors in a K_k-free coloring of G

Results: (Krawczyk, W, 2014)

- clean rectangle overlap graphs: $\chi_3 = O_\omega(1)$
- clean subtree overlap graphs & string graphs:
  construction with $\omega = k$ and $\chi_k = \Theta_k(\log \log n)$

Problem: Is χ_3 bounded in terms of ω for segment graphs/1-string graphs?

Quasi-planar graphs

Drawings of graphs in the plane:

- geometric — edges drawn as straight-line segments
- simple topological — edges drawn as 1-intersecting curves
- topological — edges drawn as arbitrary curves
- k-quasi-planar — no k pairwise crossing edges

Conjecture (Pach, Shahrokhi, Szegedy, 1996)

Every k-quasi-planar topological graph on n vertices has $O_k(n)$ edges.

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Conjecture (Pach, Shahrokhi, Szegedy, 1996)

Every k-quasi-planar topological graph on n vertices has $O_k(n)$ edges.

- true for $k = 2$ (planar graphs)
- true for $k = 3$ (Agarwal et al., 1997; Pach, Radoičić, Toth, 2006)
- true for $k = 4$ (Ackerman, 2009)

Upper bounds:

- geometric graphs: $O_k(n \log n)$ (Valtr, 1998)
- simple topological graphs: $O_k(n \log n)$ (Suk, W, 2013)
- topological graphs: $n(\log n)^{O(\log k)}$ (Fox, Pach, 2014)

Quasi-planar graphs

Conjecture

Every k-quasi-planar topological graph on n vertices has $O_k(n)$ edges.

How to guarantee $O_k(n)$ edges?

- Find a proper coloring of the edges using $O_k(1)$ colors.
  No such bound possible for segment graphs.
- Find an independent set of edges of size $\Omega_k(n)$.
  No such bound possible for segment graphs. (W, 2015)
- Find a K_3-free (K_4-free) coloring of the using $O_k(1)$ colors.
  No such bound possible for string graphs.
  But is it possible for segment graphs or 1-string graphs?

Summary of problems

1. Improve the bounding functions for rectangle graphs/circle graphs.
   3ω ? $O(\omega^2)$ / $\Omega(\omega \log \omega)$? $O(2^\omega)$
2. Improve the asymptotic bounding functions (in terms of n) for non-clean rectangle/subtree overlap graphs.
   $\Omega(\log \log n)$? $O((\log \log n)^{\omega-1})$
   / $\Omega((\log \log n)^{\omega-1})$? $O((\log \log n)^{\binom{\omega}{2}})$
3. Improve the asymptotic bounding functions (in terms of n) for segment graphs/string graphs.
   $\Omega(\log \log n)$? $O(\log n)$ / $\Omega((\log \log n)^{\omega-1})$? $(\log n)^{O(\log \omega)}$
4. Does every K_k-free 1-string graph with large χ contain an induced subgraph with radius 2 and with large χ?
5. Are intersection graphs of pseudo-parabolas χ-bounded?
6. Does every triangle-free segment intersection graph with chromatic number k contain a canonical construction of a graph with chromatic number $\Theta(k)$ as an induced subgraph?
7. Is χ_3 bounded in terms of ω for segment graphs/1-string graphs?